Amalgamation constructions

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This article is based on notes for talks I gave to the Oxford junior logic seminar on 8th March and 3rd May 2005. The early part is based on the account in Wilfred Hodges' book "Model Theory", [2].

1 Structures

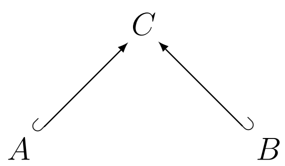
Let $\mathcal{L}$ be a countable language and M an $\mathcal{L}$-structure.

Definition 1.1. The *age* of M is the set $\mathcal{K}$ of finitely generated $\mathcal{L}$-substructures of M (up to isomorphism).

Note that the age of M has two particular properties:

Hereditary Property (HP) If $A \in \mathcal{K}$ and B is a finitely generated $\mathcal{L}$-substructure of A then $B \in \mathcal{K}$.

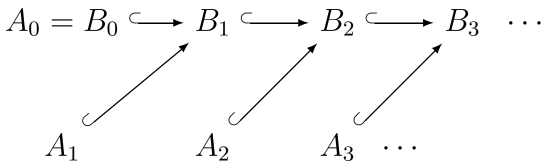
Joint Embedding Property (JEP) If $A, B \in \mathcal{K}$ then there is $C \in \mathcal{K}$ and embeddings $A \hookrightarrow C$ and $B \hookrightarrow C$.



The idea of amalgamation constructions is that we want to reconstruct an $\mathcal{L}$-structure from its finitely generated substructures, or construct a new $\mathcal{L}$-structure from some set of finitely generated structures.

Theorem 1.2. *Any countable class $\mathcal{K}$ of finitely generated $\mathcal{L}$-structures with HP and JEP is the age of some countable $\mathcal{L}$-structure.*

Proof. Enumerate $\mathcal{K}$ as $(A_n)_{n\in\mathbb{N}}$. We construct a chain $(B_n)_{n\in\mathbb{N}}$ inductively as follows. Set $B_0 = A_0$. Then for $n \in \mathbb{N}$, choose B_{n+1} from $\mathcal{K}$ such that B_n and A_{n+1} embed in it. We get the following.



Take M to be the union of the chain $(B_n)_{n\in\mathbb{N}}$. Then every A_n embeds into M, so $\mathcal{K}$ is contained in the age of M. Suppose that C is a finitely generated substructure of M, with generators $c_1, \ldots, c_m$. Then there is $n \in \mathbb{N}$ such that all the c_i lie in some B_n, so C is a finitely generated substructure of B_n, and so is in $\mathcal{K}$ by HP. □

Note that we only use HP to show that the age of M is exactly $\mathcal{K}$, so we don't actually need this to do the construction.

Example 1.3. $\langle \mathbb{Q}; \leqslant \rangle$ and $\langle \mathbb{N}; \leqslant \rangle$ have the same age. Indeed, every infinite linear order has the same age.

From this we see that our construction was not at all canonical. We now fix this.

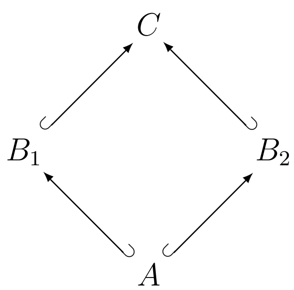
Definition 1.4. A (countable) structure M is $\aleph_0$-homogeneous iff for any finitely generated substructures A_1, A_2 of M, any isomorphism $\sigma : A_1 \xrightarrow{\cong} A_2$ extends to an automorphism of M.

Note that $\langle \mathbb{Q}; \leqslant \rangle$ is homogeneous but $\langle \mathbb{Z}; \leqslant \rangle$ is not. For example the map

$$\begin{array}{rcl} \{1,2\} & \to & \{1,3\} \\ 1 & \mapsto & 1 \\ 2 & \mapsto & 3 \end{array}$$

does not extend to an automorphism of $\langle \mathbb{Z}; \leqslant \rangle$.

Definition 1.5. A set of $\mathcal{L}$-structure $\mathcal{K}$ has the *Amalgamation Property* (AP) iff given $A, B_1, B_2 \in \mathcal{K}$ and embeddings $A \hookrightarrow B_i$ we can complete the square



with some $C \in \mathcal{K}$.

Theorem 1.6 (Fraissé Amalgamation Theorem). *Let $\mathcal{L}$ be a countable language and $\mathcal{K}$ a nonempty countable set of finitely generated $\mathcal{L}$-structures with HP, JEP and AP. Then there is a unique countable homogeneous $\mathcal{L}$-structure M with age $\mathcal{K}$.*

Proof. Uniqueness is by the back and forth method using homogeneity. For existence, we construct a sequence $(B_n)_{n\in\mathbb{N}}$ as before but taking more care about the order. See [2] or [1] for more details. □

This process is known as an *amalgamation construction* and M is called the *amalgam* or *limit structure* of $\mathcal{K}$.

2 Theories

We have seen how to use amalgamation constructions to produce (homogeneous) countable structures. One reason for doing this is to produce complete first order theories, the theory of the amalgam. To have any hope of a general procedure to axiomatize the amalgam M we need to have some conditions on $\mathcal{K}$, so we will assume here that $\mathcal{K}$ is the set of finitely generated models of some incomplete theory T. Since $\mathcal{K}$ has HP, T must be a universal theory. Indeed, $T = \mathrm{Th}_\forall M$.

In some situations, axiomatizing T is now easy. For example, if L has no function symbols and is a finite language (e.g. for graphs, partial or total orders), or more generally if every finitely generated structure is finite (boolean algebras) then for each $A \hookrightarrow B$ in $\mathcal{K}$, let $\varphi_A(x)$ be a formula expressing that x has isomorphism type A, $\theta_{AB}(x, y)$ be a formula expressing that (x, y) has isomorphism type B, then T together with all sentences of the form

$$\forall x[\varphi_A(x) \to \exists y[\theta_{AB}(x, y)]]$$

is a complete axiomatization of $\mathrm{Th}\, M$. In general it is not so easy to write down an axiomatization.

3 A Generalization

Suppose we want to construct a different structure with a given age – not the homogeneous one. We can do this by amalgamating over a subset of all possible embeddings. Provided that the category "$\mathcal{K}$ with the chosen embeddings" has JEP and AP, we still produce a unique countable structure, homogeneous with respect to the chosen embeddings. The construction and uniqueness proof are the same as for Fraissé's theorem.

Example 3.1. Take $\mathcal{K}$ to be the set of finite linear orders (the age of any infinite linear order), and say an embedding $A \hookrightarrow B$ is allowed iff for each $a_1, a_2 \in A$ and $b \in B$, if $B \models a_1 < b < a_2$ then $b \in A$.

The amalgam of $\mathcal{K}$ with these allowed embeddings is $\langle \mathbb{Z}; \leqslant \rangle$.

At this level of generality we have to specify both a class of structures and which embeddings are "allowed". The conditions on the allowed embeddings apart from JEP and AP are that they should form a category, that is the identity embedding for each structure should be allowed and the composition of two allowed embeddings should be allowed, and that we should be able to take unions of chains in the appropriate way. In principle this gives us a way of formulating the JEP and AP on many categories other than those of L-structures and embeddings, and producing amalgams, but we don't do this here. [1] considers this in a high level of generality with some applications to group theory and domain theory. We will use the following more specific version of their theorem.

Theorem 3.2. *Let $\mathcal{L}$ be a countable language, $\mathcal{K}$ a category of countable $\mathcal{L}$-structures together with certain chosen embeddings and $\mathcal{K}_0$ the full subcategory of finitely generated $\mathcal{L}$-structures. Suppose the following.*

- *if $A_1 \hookrightarrow A_2 \hookrightarrow A_3 \hookrightarrow \cdots$ is an ω-chain in $\mathcal{K}_0$ and A is the union of the chain then A and the embeddings $A_n \hookrightarrow A$ are in $\mathcal{K}$.*
- *Every $A \in \mathcal{K}$ is the union of some ω-chain in $\mathcal{K}_0$.*
- *$\mathcal{K}_0$ contains only countably many objects (up to isomorphism).*

Then $\mathcal{K}$ contains a $\mathcal{K}$-universal and $\mathcal{K}_0$-homogeneous object iff $\mathcal{K}_0$ has JEP and AP. If such an object exists it is unique up to isomorphism.

4 Amalgamation with predimension

4.1 Forests

The amalgamation method used by Hrushovski in [3] to construct his strongly minimal sets is a special case of the general method described above. He wanted to construct structures with a notion of dimension, which he did by choosing the allowed embeddings by means of a predimension function, so called because the dimension notion in the amalgam arises from it.

The construction of the strongly minimal sets actually has two parts: the amalgamation which produces a structure of infinite Morley Rank and a second part which "collapses" this structure to a finite Morley Rank (indeed strongly minimal) structure. We consider only the amalgamation part here.

Before giving the general method, we illustrate it with a simple example which contains the main ideas of the amalgamation construction and how the dimension theory arises.

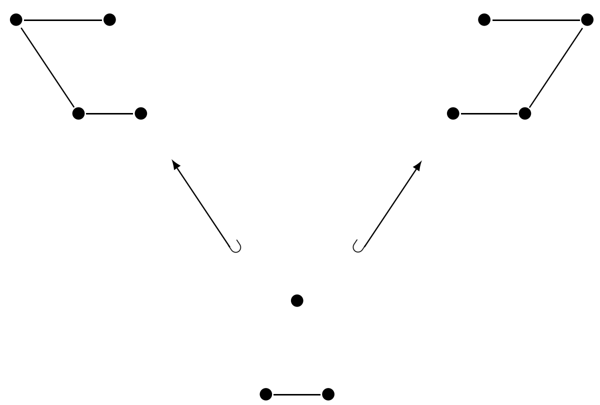
Let $\mathcal{G}$ be the collection of all graphs (considered as sets with an irreflexive symmetric binary predicate). For a finite graph G, define its predimension $\delta(G) = |G| - |e(G)|$, the number of vertices minus the number of edges. Define

$$\mathcal{F} = \{G \in \mathcal{G} \mid (\forall X \subseteq_{\mathrm{fin}} G)\delta(X) \geqslant 1\}$$

and define $\mathcal{K}$ to be the collection of finite graphs in $\mathcal{F}$.

It is easy to see that $\mathcal{F}$ is the collection of *acyclic* graphs (or forests, each being a disjoint union of trees), and that for $X \in \mathcal{K}$, the predimension $\delta(X)$ is simply the number of connected components of X.

The category of $\mathcal{K}$ with all embeddings does not have AP. For example, any amalgam of the following diagram would have to contain a 5-cycle.



However, we can choose certain embeddings which respect the predimension and get a category which does have AP. We call these embeddings *strong* embeddings.

If $G, H \in \mathcal{F}$, we define an embedding $G \hookrightarrow H$ to be *strong* iff distinct components of G remain distinct in H. In terms of the predimension, this is true iff

$$(\forall X \subseteq_{\mathrm{fin}} H)[\delta(X \cap G) \leqslant \delta(X)].$$

We write a strong embedding as $G \lhd H$. We consider $\mathcal{F}$ with these strong embeddings as a category, and $\mathcal{K}$ as a subcategory. It is easy to check that this category $\mathcal{K}$ has AP and JEP, and so our theorem tells us that we have a unique, homogeneous universal countable forest, F.

We can actually write down the universal theory of $\mathcal{K}$ as the theory of graphs together with a set of sentences ruling out cycles of each finite length. Using this we can even give a complete axiomatization for $\operatorname{Th} F$ by the method described earlier, which works just as well even though we are not considering all embeddings.

For this particular example however we can give another description of F as a countable acyclic graph where the degree of each vertex is infinite and there are infinitely many connected components. Any two infinite acylic graphs with infinite degree of every vertex are elementarily equivalent (as can be seen by a back and forth argument), so this gives another axiomatization of $\operatorname{Th} F$.

Dimension Theory For any finite subset $X \subseteq F$, define $d(X)$ to be the number of connected components of F which X meets. This is given in terms of the predimension by

$$d(X) = \min\{\delta(Y) \mid X \subseteq Y \text{ and } Y \subseteq_{f.g.} F\}.$$

We can now define a notion of independence with the exchange property which leads to a notion of basis, and thus extend d to any subset of F. In this example we see that $d(X)$ still coincides with the cardinality of the set of connected components which meet X.

This procedure is very general and although we can understand forests and the theory of the universal forest without the amalgamation method, there are many examples for which this is a useful tool.

4.2 A general procedure

We now describe the procedure in general terms and give some more examples. Let $\mathcal{L}$ be a countable language, $\mathcal{K}$ a class of countable $\mathcal{L}$-structures and $\mathcal{K}_0$ the subclass of finitely generated $\mathcal{L}$-structures. If $A \subseteq B$ with $A \in \mathcal{K}_0$ and $B \in \mathcal{K}$ we write $A \subseteq_{f.g.} B$. Suppose we also have a "predimension" function $\mathcal{K}_0 \xrightarrow{\delta} \mathbb{N}$ such that if $A \cong B$ then $\delta(A) = \delta(B)$.

For $A, B \in \mathcal{K}$ we define an embedding $A \hookrightarrow B$ to be *strong* iff

$$(\forall X \subseteq_{f.g.} B)[\delta(X \cap A) \leqslant \delta(X)]$$

and we write a strong embedding as $A \lhd B$. Note that identity embeddings are strong and the composite of strong embeddings is strong, so we may think of $\mathcal{K}$ and $\mathcal{K}_0$ with these strong embeddings as categories.

We wish to apply theorem 3.2, so we assume that $\mathcal{K}$ is precisely the category of unions of ω-chains in $\mathcal{K}_0$, that $\mathcal{K}_0$ contains only countably many objects up to isomorphism and that $\mathcal{K}_0$ has JEP and AP. The definition of strong embeddings is local, that is it depends only on the behaviour of finitely generated parts of the objects, so it follows that if

$$A_1 \lhd A_2 \lhd A_3 \lhd \cdots$$

is a chain in $\mathcal{K}_0$ with union A then the embeddings $A_i \hookrightarrow A$ are also strong.

Applying theorem 3.2 we see that $\mathcal{K}$ contains a $\mathcal{K}$-universal and $\mathcal{K}_0$-homogeneous object which is unique up to isomorphism. Call this countable structure M.

Examples 4.1. The first three examples are the familiar examples of theories with a good dimension notion.

Sets Take $\mathcal{L}$ to be the empty language and $\mathcal{K}$ to be the class of countable sets. $\mathcal{K}_0$ is then the class of finite sets. Take $\delta(X) = |X|$. Then every embedding is strong, and the amalgam M is just a countable set. Obviously we don't need to use amalgamation to produce this!

Vector spaces Take $\mathcal{L}$ to be the language of vector spaces over some countable division ring R, and $\mathcal{K}$ the class of countable R-vector spaces. $\mathcal{K}_0$ is the class of finite-dimensional vector spaces, and we define $\delta(V) = \dim V$. The amalgam is the vector space of dimension $\aleph_0$.

It is clear that this procedure would work in the same way for uncountable division rings, so we have not given the most general form of the construction.

Fields Take $\mathcal{L}$ to be the language of rings, and $\mathcal{K}$ the class of countable integral domains of fixed characteristic p (possibly zero). Each finitely generated ring has finite transcendence degree, so we can define $\delta(R) = \operatorname{td} R$. The amalgam will be the algebraically closed field of characteristic p and transcendence degree $\aleph_0$.

These are the three "basic" dimension notions of the trichotomy: combinatorial, linear and algebraic. The examples given by Hrushovski involve combining these dimension notions in some way.

The ab initio example Take $\mathcal{L}$ to have a single n-ary relation symbol, R. (Usually $n = 3$). For any finite $\mathcal{L}$-structure X, define $\delta(X) = |X| - |R \cap X^n|$. Take $\mathcal{K}$ to be the class of countable $\mathcal{L}$-structures A such that for every $X \subseteq_{\text{fin}} A$, $\delta(X) \geqslant 0$. Here the dimension notion is obtained by subtracting one basic dimension from another.

Other relational examples We can do similar constructions on a field or vector space where we add a relation symbol (or several) to the field or vector space language. The predimension is then given for example by $\delta(X) = \mathrm{td}(X) - |R \cap X^n|$. We take $\mathcal{K}$ as above. We can even add a countable family R_i of relation symbols and take $\mathcal{K}$ to be the class of structures where $\delta(X) = \mathrm{td}(X) - \sum_i |R_i \cap X^{n_i}|$ is hereditarily non-negative.

Examples with functions Instead of adding a relation symbol we can add a function symbol. Adding a unary function symbol f to a field we can define the predimension as $\delta(X) = \mathrm{td}(X \cup f(X)) - |X|$. (If instead of adding the function we add its graph as a relation, the function and relation notions of predimension agree. So this isn't really different from the relational examples.) The amalgam in this example is an algebraically closed field with a "generic" function. Wilkie and Koiran showed that there are complex functions (Liouville functions) which have this theory.

Pseudo-exponentiation A more complicated construction along these lines is Zilber's pseudo-exponentiation. Again we have a unary function ex on a field, but this time the function is constrained to be a homomorphism from the additive group to the multiplicative group. The additive group is a $\mathbb{Q}$-vector space, and so it makes sense to define the predimension as $\delta(X) = \mathrm{td}(X \cup \mathrm{ex}(X)) - \dim_{\mathbb{Q}}(X)$. The class $\mathcal{K}$ is taken to be those structures which have hereditarily non-negative predimension and where the homomorphism ex has a "standard kernel".

Fusion If $\mathcal{L}_1$ and $\mathcal{L}_2$ are relational languages and δ_1, δ_2 are dimension notions on classes $\mathcal{K}_i$ of $\mathcal{L}_i$-structures then Hrushovski's fusion of the two classes is given by taking $\mathcal{L} = \mathcal{L}_1 \sqcup \mathcal{L}_2$ and $\delta(X) = \delta_1(X) + \delta_2(X) - |X|$. In this way he produced examples such as a strongly minimal set with two different field structures on it.

In all of these examples, the predimension function δ is given by some $\mathbb{Z}$-linear combination of basic dimension notions, where any dimension notion with negative coefficient is one of the first two, not transcendence degree. In these situations, proving that the category $\mathcal{K}_0$ has the amalgamation property is essentially the same calculation as for the example of acyclic graphs. Checking the other conditions such as JEP is much easier.

4.3 Dimension theory

We have the notion of predimension and now wish to use it to define our notion of dimension in M. Note that for every $X \subseteq_{f.g.} M$, there is some finitely generated Y extending X such that $Y \lhd M$, since M is the union of a chain

$$A_1 \lhd A_2 \lhd A_3 \lhd \cdots$$

of finitely generated structures, so X must be contained in some A_i. Since the predimension is constrained to lie in $\mathbb{N}$ which is well-ordered, the following definition of the dimension $d(X)$ of X is well-defined.

$$d(X) = \min\{\delta(Y) \mid X \subseteq Y \text{ and } Y \subseteq_{f.g.} M\}.$$

Note that if $X, Y \subseteq_{f.g.} M$ and $X \subseteq Y$ then $d(X) \leqslant d(Y)$. Also $X \lhd M$ iff $d(X) = \delta(X)$, so any Y realising the minimum is strong in M.

We define the *relative dimension* of X/A where $X \subseteq_{f.g.} M$ and $A \subseteq M$ by

$$d(X/A) = \min\{d(XY) - d(Y) \mid Y \subseteq_{f.g.} A\}$$

where XY is the structure generated by $X \cup Y$. Since $d(XY) \geqslant d(X)$, we always have $d(X/A) \geqslant 0$.

Until now we have always assumed that $\subseteq$ meant a substructure, not any subset, but we can extend the predimension, dimension and relative dimension notions to subsets by defining for example $d(X/A) = d(\langle X \rangle / \langle A \rangle)$ where $\langle X \rangle$ is the substructure of M generated by X.

This d will give the dimension of finite dimensional sets, but a dimension theory should extend to infinite dimensional sets too. To do this we construct a pregeometry from d, and use that to define dimension more generally. Define an operator $\mathrm{cl} : \mathbb{P}M \to \mathbb{P}M$ by

$$x \in \mathrm{cl}(A) \iff d(x/A) = 0$$

for any singleton $x \in M$ and any subset $A \subseteq M$. We call $\mathrm{cl}(A)$ the *closure* of A and also write $x \in \mathrm{cl}(A)$ for a tuple x if every coordinate of x is in the closure of A.

In all the examples listed above this operator is a pregeometry, that is it satisfies the following five conditions.

1. $A \subseteq \mathrm{cl}(A)$
2. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$
3. If $A \subseteq B$ then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$

4. If $x \in \mathrm{cl}(A)$ then there is a finite tuple y in A such that $x \in \mathrm{cl}(y)$.

5. If x, y are singletons and $x \in \mathrm{cl}(Ay)$ then either $x \in \mathrm{cl}(A)$ or $y \in \mathrm{cl}(Ax)$.

The first three conditions say that cl is a closure operator, the fourth is called *finite character*, and the fifth is the *exchange* principle, a generalization of the Steinitz exchange principle of linear algebra. From a pregeometry one immediately gets notions of independence, spanning set and basis, and the dimension of a set is the (well-defined) cardinality of a basis for that set. Where this dimension is finite, it agrees with the function d. This procedure is described for example at the start of chapter 8 of [4].

In general we would need to make further assumptions on the nature of the predimension function to be able to deduce that cl is a pregeometry. The known examples are all variations of the examples I have given, and in these cases the proof goes by observing that we can make some stronger statements about witnesses for the amalgamation property.

In some examples this dimension notion will extend well to all other models of $\mathrm{Th}\, M$, but we have placed so few restrictions on our construction that we cannot hope for this in general. For example, the dimension notion for pseudo-exponentiation extends naturally to some non-elementary proper subclass of the elementary class of the amalgam.

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